

# Finite-state self-similar actions of nilpotent groups

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## Abstract

Let  $G$  be a finitely generated torsion-free nilpotent group and  $\phi : H \rightarrow G$  be a surjective homomorphism from a subgroup  $H < G$  of finite index with trivial  $\phi$ -core. For every choice of coset representatives of  $H$  in  $G$  there is a faithful self-similar action of the group  $G$  associated with  $(G, \phi)$ . We characterize the existence of finite-state self-similar actions for  $(G, \phi)$  in terms of the Jordan normal form of  $\phi$  viewed as an automorphism of the Lie algebra of  $G$ .

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## 1 Introduction and Preliminaries

An action of a group  $G$  on the set  $X^*$  of words over a finite alphabet  $X$  is called *self-similar* if for every  $x \in X$  and  $g \in G$  there exist  $y \in X$  and  $h \in G$  such that  $g(xv) = yh(v)$  for all words  $v \in X^*$ . Self-similar group actions appear naturally in many areas of mathematics and have applications to holomorphic dynamics, fractal geometry, combinatorics, automata theory, etc. (see [7] and the reference therein).

The theory of self-similar group actions can be regarded as the study of positional numeration systems on groups. Bases in these number systems are virtual endomorphisms of groups. A *virtual endomorphism* of a group  $G$  is a homomorphism  $\phi : H \rightarrow G$  from a subgroup  $H < G$  of finite index to  $G$ . To produce a self-similar action of the group  $G$  with “base”  $\phi$  we need to choose a set  $D$  of coset representatives of  $H$  in  $G$  called a *digit set*. Let us enumerate the elements of  $D$  by the letters of an alphabet  $X$  (here  $|X| = [G : H]$  and  $D = \{h_x, x \in X\}$ ). The *self-similar action* of  $G$  on the space  $X^*$  *associated to the triple*  $(G, \phi, D)$  is constructed as follows. Every element of the group stabilizes the empty word. For every  $x \in X$  and  $v \in X^*$  the action of an element  $g \in G$  is defined recursively by the rule

$$g(xv) = yh(v) \quad \text{with} \quad h = \phi(h_y^{-1}gh_x), \quad (1)$$

where  $y \in X$  is the unique letter such that  $h_y^{-1}gh_x \in H$ . The constructed action may be not faithful. The kernel of the action does not depend on the choice of the set  $D$  and is equal to the maximal normal  $\phi$ -invariant subgroup of  $G$  called the  $\phi$ -core ([7, Proposition 2.7.5]).

Conversely, one can associate a virtual endomorphism with every faithful self-similar action  $(G, X^*)$  as follows. The element  $h$  from the definition of self-similar action is called the *state* of  $g$  at  $x$  and is denoted by  $g|_x$  (this element is unique if the action is faithful); iteratively we define the state of  $g$  at every word by the rule  $g|_{x_1x_2\dots x_n} = g|_{x_1}|_{x_2}\dots|_{x_n}$ . For every letter  $x \in X$  the stabilizer  $St_G(x)$  has finite index in  $G$  and then the map  $\phi_x : St_G(x) \rightarrow G$  given by  $\phi_x(g) = g|_x$  is a virtual endomorphism of  $G$ . If in addition the action  $(G, X^*)$  is *self-replicating* (recurrent), i.e.,  $\phi_x$  is surjective and  $G$  acts transitively on  $X$ , then  $(G, X^*)$  can be given by the triple  $(G, \phi_x, D)$  for some choice of the digit set  $D$ .

As a simple example, consider the group  $\mathbb{Z}$  with the homomorphism  $\phi : 2\mathbb{Z} \rightarrow \mathbb{Z}$ ,  $\phi(2a) = a$ , and choose the digit set  $D = \{0, 1\}$ , which is also used as an alphabet with a slight abuse in notations. The associated self-similar action corresponds to the binary number system on  $\mathbb{Z}$ . We have  $a|_{x_1x_2\dots x_n} = b$  and  $a(x_1x_2\dots x_n) = y_1y_2\dots y_n$  for  $a, b \in \mathbb{Z}$  if and only if

$$a = (y_1 - x_1) + 2(y_2 - x_2) + 2^2(y_3 - x_3) + \dots + 2^n(y_n - x_n) + 2^nb.$$

In particular, if  $a|_{00\dots 0} = 0$  then the image  $a(00\dots 0)$  is the usual binary expansion of  $a$ .

Self-similar group actions are closely related to groups generated by automata. Groups generated by finite automata correspond to finite-state self-similar actions of finitely generated groups. Recall that a faithful self-similar action  $(G, X^*)$  is called *finite-state* if for every  $g \in G$  the set of its states  $\{g|_v : v \in X^*\}$  is finite. Then a finitely generated group has a faithful finite-state self-similar action if and only if it can be generated by a finite automaton. The fundamental question in this theory is what groups possess finite-state self-similar actions, i.e., can be realized by finite automata. This property was proved for free abelian groups  $\mathbb{Z}^n$  [8], Grigorchuk group [5],  $GL_n(\mathbb{Z})$  [3], lamplighter groups [10], free groups and free products of cyclic groups of order 2 [9], Baumslag-Solitar groups  $B(1, m)$  [1], certain nilpotent groups [2], etc.

The finite-state self-similar actions of  $\mathbb{Z}^n$  can be characterized in term of the associated virtual endomorphism as shown by Nekrashevych and Sidki in [8] (see also [7, Theorem 2.12.1]). A virtual endomorphism of  $\mathbb{Z}^n$  is uniquely extended to a linear operator of  $\mathbb{R}^n$ . Then a faithful self-replicating self-similar action of  $\mathbb{Z}^n$  with virtual endomorphism  $\phi$  is finite-state if and only if the spectral radius of  $\phi$  is less than 1. In particular, there is no dependence on the choice of coset representatives.

In this paper we consider self-similar actions of finitely generated torsion-free nilpotent groups. The main goal is to generalize the above mentioned result of Nekrashevych and Sidki. However, self-similar actions of nilpotent groups have a new level of complexity comparing to the actions of abelian groups. For example, a nilpotent group with fixed virtual endomorphism may have a faithful finite-state self-similar action for one choice of coset representatives and be not finite-state for another choice (see example with Heisenberg group in Section 3). Hence we need to answer two questions: Under what conditions on  $\phi$

does there exist a finite-state action for  $(G, \phi)$ ? Under what conditions does every action for  $(G, \phi)$  finite-state?

Let  $G$  be a finitely generated torsion-free nilpotent group with surjective virtual endomorphism  $\phi : H \rightarrow G$ . Since we are interested in faithful self-similar actions of the group  $G$ , we assume that  $\phi$ -core is trivial. Then Corollary 1 in [2] implies that  $\phi$  is also injective and hence it is an isomorphism. The same corollary says that if we know that  $\phi$  is an isomorphism then  $\phi$ -core is trivial if and only if the virtual endomorphism  $\phi|_{Z(H)} : Z(H) \rightarrow Z(G)$  of the center  $Z(G)$  has trivial core. Since  $Z(G)$  is abelian, by [7, Proposition 2.9.2] the  $\phi|_{Z(H)}$ -core is trivial if and only if no eigenvalue of  $\phi|_{Z(H)}$  is an algebraic integer. Hence one can effectively check when  $\phi$ -core is trivial. By a theorem of Mal'cev (see [6]) there exists the unique real nilpotent Lie group  $L$ , *Mal'cev completion* of  $G$ , such that the group  $G$  is a discrete subgroup of  $L$  and the topological space  $L/G$  is compact. Since  $H$  is a subgroup of finite index, the isomorphism  $\phi : H \rightarrow G$  lifts to an automorphism of the Lie group  $L$  also denoted by  $\phi$ . Let  $\mathcal{L}$  be the Lie algebra of  $L$  and denote the automorphism of  $\mathcal{L}$  induced by  $\phi$  also by  $\phi$ . Then the existence of finite-state self-similar action of the group  $G$  can be characterized in terms of the Jordan normal form of  $\phi$ .

**Theorem 1.** *Let  $G$  be a finitely generated torsion-free nilpotent group. Let  $(G, X^*)$  be a faithful self-replicating self-similar action with virtual endomorphism  $\phi$  (associated to some letter  $x \in X$ ). If the action  $(G, X^*)$  is finite-state then the spectral radius of  $\phi$  is not greater than 1 and for every eigenvalue of modulus 1 the associated Jordan cells in the Jordan normal form of  $\phi$  have size 1. Conversely, if the virtual endomorphism  $\phi$  satisfies the previous condition then there exists a finite-state self-similar action of  $G$  with virtual endomorphism  $\phi$ .*

One can restate the theorem as follows. Let  $\phi$  be a surjective virtual endomorphism of  $G$  with trivial core. There exists a digit set  $D$  with  $e \in D$  such that the self-similar action associated to  $(G, \phi, D)$  is finite-state if and only if the Jordan normal form of  $\phi$  satisfies the condition in the theorem.

**Theorem 2.** *Let  $G$  be a finitely generated torsion-free nilpotent group, and let  $\phi$  be a surjective virtual endomorphism of  $G$  with trivial core. Every self-similar action of  $(G, \phi)$  is finite-state if and only if the spectral radius of  $\phi$  is less than 1.*

In particular, if the Jordan normal form of  $\phi$  satisfies the condition in Theorem 1 and  $\phi$  has an eigenvalue of modulus 1, then the pair  $(G, \phi)$  possesses both finite-state and non-finite-state self-similar actions. This situation cannot happen for abelian group  $\mathbb{Z}^n$ , because if a virtual endomorphism of  $\mathbb{Z}^n$  has an eigenvalue of modulus 1 then it has a non-trivial core.

## 2 Proof of Theorems 1 and 2

Recall that there exists a bijection between the Lie algebra  $\mathcal{L}$  and the Lie group  $L$  given by the exponential map  $\exp : \mathcal{L} \rightarrow L$  with inverse  $\log : L \rightarrow \mathcal{L}$ . The automorphism  $\phi$  of  $L$

and  $\mathcal{L}$  satisfies

$$\phi(\log(g)) = \log(\phi(g)) \text{ for all } g \in L. \quad (2)$$

Let  $\mathcal{L}_{\mathbb{Q}} \subset \mathcal{L}$  be the set of all linear combinations of vectors from  $\log(G)$  over  $\mathbb{Q}$ . By Theorem 5.1.8 (a) in [4]  $\mathcal{L}_{\mathbb{Q}}$  is a Lie algebra over  $\mathbb{Q}$  such that  $\mathcal{L}_{\mathbb{Q}} \otimes \mathbb{R} = \mathcal{L}$ . It is usually said that this defines a rational structure on  $L$ . It is easy to see that  $\phi(\mathcal{L}_{\mathbb{Q}}) \subset \mathcal{L}_{\mathbb{Q}}$ . Indeed, since  $H$  is of finite index in  $G$  it is also a uniform subgroup of  $L$ , which is by definition commensurable with  $G$ . Thus by Theorem 5.1.12 in [4] the Lie algebra  $\mathcal{L}_{\mathbb{Q}}$  is also equal to the  $\mathbb{Q}$ -span of vectors from  $\log(H)$ . Since  $\phi(H) \subset G$  we have that  $\phi(\mathcal{L}_{\mathbb{Q}}) \subset \mathcal{L}_{\mathbb{Q}}$  by Equation (2). In particular it follows that the matrix of  $\phi$  has rational entries in any basis of  $L_{\mathbb{Q}}$ . Moreover it can be shown using induction on the length of lower central series of  $L$  that  $[G : H] = \det \phi^{-1}$ .

A self-similar action  $(G, X^*)$  is called *contracting* if there exists a finite set  $\mathcal{N} \subset G$  with the property that for every  $g \in G$  there exists  $n \in \mathbb{N}$  such that  $g|_v \in \mathcal{N}$  for all words  $v \in X^*$  of length  $\geq n$ . Every contracting action is finite-state by definition. A self-replicating self-similar action is contracting if and only if the associated virtual endomorphism has spectral radius less than 1 (see [7, Proposition 2.11.11]).

*Proof of sufficiency in Theorem 1.* The assumption on the Jordan normal form of  $\phi$  implies the following crucial property: for every  $g \in L$  the sequence  $\phi^n(g)$  is bounded (i.e., belongs to a compact set).

The Lie algebra  $\mathcal{L}$  decomposes in the direct sum  $\mathcal{L} = \mathcal{L}_c \oplus \mathcal{L}_r$ , where  $\mathcal{L}_c$  is a  $\phi$ -invariant subalgebra such that  $\phi|_{\mathcal{L}_c}$  has spectral radius less than 1 (contracting), and the spectrum of  $\phi|_{\mathcal{L}/\mathcal{L}_c}$  consists only of numbers of modulus 1. Consider the  $\phi$ -invariant subgroup  $L_c = \exp(\mathcal{L}_c)$  of the Lie group  $L$  that corresponds to the subalgebra  $\mathcal{L}_c$ . One can define  $L_c$  directly as a subgroup of all  $g \in L$  such that  $\phi^n(g) \rightarrow 1$  as  $n \rightarrow \infty$ . Define the group  $G_c = G \cap L_c$  and its subgroup  $H_c = H \cap L_c = \phi^{-1}(G_c) < G_c$  of finite index  $[G_c : H_c] = \det(\phi|_{\mathcal{L}_c})^{-1}$ . Then  $\phi|_{H_c} : H_c \rightarrow G_c$  is a contracting isomorphism and every self-similar action of  $(G_c, \phi|_{H_c})$  is contracting.

Notice that  $\det(\phi|_{\mathcal{L}/\mathcal{L}_c})$  is a positive number as since  $\det(\phi) = \det(\phi|_{\mathcal{L}/\mathcal{L}_c}) \det(\phi|_{\mathcal{L}_c})$ , and, at the same time, it is a product of numbers of modulus 1. Hence  $\det(\phi|_{\mathcal{L}/\mathcal{L}_c}) = 1$  and we get  $[G : H] = \det(\phi)^{-1} = \det(\phi|_{\mathcal{L}_c})^{-1} = [G_c : H_c]$ .

Take any coset representatives  $h_1, h_2, \dots, h_d$  for  $H_c$  in  $G_c$ . Since  $H \cap G_c = H_c$  and  $[G : H] = [G_c : H_c]$ , the elements  $h_1, h_2, \dots, h_d$  are also coset representatives of  $H$  in  $G$ . Let us consider the associated self-similar action  $(G, X^*)$  given by Equation (1). Take any element  $g \in G$  and for every word  $x_1 x_2 \dots x_n \in X^*$  consider the state

$$\begin{aligned} g|_{x_1 x_2 \dots x_n} &= \phi(h_{y_n}^{-1} \dots \phi(h_{y_2}^{-1} \phi(h_{y_1}^{-1} g h_{x_1}) h_{x_2}) \dots h_{x_n}) \\ &= \phi(h_{y_n}^{-1}) \dots \phi^{n-1}(h_{y_2}^{-1}) \phi^n(h_{y_1}^{-1}) \phi^n(g) \phi^n(h_{x_1}) \phi^{n-1}(h_{x_2}) \dots \phi(h_{x_n}), \end{aligned} \quad (3)$$

where  $y_1 y_2 \dots y_n = g(x_1 x_2 \dots x_n)$ . The sequence  $\phi^n(g)$  is bounded in  $L$ . The elements  $h_{x_1}, h_{x_2}, \dots, h_{x_n}$  are taken from a finite subset of the  $\phi$ -invariant subgroup  $L_c$  on which  $\phi$  is contracting. Then the set of all products of the form  $\phi^n(h_{x_1}) \phi^{n-1}(h_{x_2}) \dots \phi(h_{x_n})$  belongs to a compact subset of  $L_c$ . Since the product in (3) belongs to the lattice  $G$ , it assumes a finite number of values. Hence the action  $(G, X^*)$  is finite-state.  $\square$

*Proof of necessity in Theorem 1.* Let  $(G, X^*)$  be a finite-state self-similar action with virtual endomorphism  $\phi$  associated to the letter  $x_1 \in X$ , i.e.,  $\phi = \phi_{x_1}$  and  $H = St_G(x_1)$ . Let  $\{h_1 = e, h_2, \dots, h_d\}$  be the corresponding set of coset representatives.

**Lemma 1.** *The eigenvalues of  $\phi$  have modulus  $\leq 1$ . Moreover, every eigenvalue of modulus 1 is a root of unity.*

*Proof.* Put  $\mathcal{L}^{(0)} = \mathcal{L}$  and let  $\mathcal{L}^{(i)} = [\mathcal{L}, \mathcal{L}^{(i-1)}]$  be the lower central series of the Lie algebra  $\mathcal{L}$ . Since  $\phi$  is an automorphism of  $\mathcal{L}$  it preserves every term  $\mathcal{L}^{(i)}$  and induces an automorphism  $\bar{\phi}_i$  on the quotient  $\mathcal{L}^{(i)}/\mathcal{L}^{(i+1)}$ . The spectrum of  $\phi$  is a union of the spectra of  $\bar{\phi}_i$ . At the same time, every linear map  $\bar{\phi}_i$  is a quotient of the tensor product  $\bar{\phi}_0 \otimes \bar{\phi}_0 \otimes \dots \otimes \bar{\phi}_0$  (see [11, Theorem 3.1]). Hence it is enough to prove the statement for the automorphism  $\bar{\phi}_0$ .

Let  $\lambda$  be an eigenvalue of  $\bar{\phi}_0$ . Take a basis of  $\mathcal{L}/[\mathcal{L}, \mathcal{L}]$  in which  $\bar{\phi}_0$  has Jordan normal form, and consider the coordinate of vectors in this basis that corresponds to an eigenvector with eigenvalue  $\lambda$ . There exists a linear map  $\xi : \mathcal{L} \rightarrow \mathbb{C}$  such that  $\xi([\mathcal{L}, \mathcal{L}]) = 0$  and  $\xi(\phi(l)) = \lambda\xi(l)$  for all  $l \in \mathcal{L}$ . We compose  $\xi$  with the logarithmic map  $\log : L \rightarrow \mathcal{L}$  and denote the composition also by  $\xi$ . Note that  $\log(g_1g_2) = \log(g_1) + \log(g_2) \bmod [L, L]$ . Thus we have a map  $\xi : L \rightarrow \mathbb{C}$  such that  $\xi(g_1g_2) = \xi(g_1) + \xi(g_2)$  and  $\xi(\phi(g)) = \lambda\xi(g)$ . The rest of the proof is very similar to the proof of Theorem 2.12.1 in [7], so we only sketch it here.

Since  $G$  is a lattice in  $L$ , there exists  $g \in G$  such that  $\xi(g) \neq 0$ . Let us consider the states  $g|_v$  for  $v \in X^*$ . By Equation (1) we have  $g|_x = \phi(h_{g(x)}^{-1}gh_x)$  for every  $x \in X$ . Then

$$\xi(g|_x) = \lambda\xi(g) + \lambda(\xi(h_x) - \xi(h_{g(x)})) = \lambda\xi(g) + d_x,$$

where  $d_x = \lambda(\xi(h_x) - \xi(h_{g(x)}))$ .

Suppose  $|\lambda| > 1$ . Since  $\sum_{x \in X} d_x = 0$ , it follows that there exists  $x_1 \in X$  such that  $|\xi(g|_{x_1})| > |\xi(g)|$ . Thus we can iteratively construct letters  $x_n \in X$  such that  $|\xi(g|_{x_1x_2\dots x_{n+1}})| > |\xi(g|_{x_1x_2\dots x_n})|$  for each  $n$ . Hence  $g$  is not finite-state, contradiction.

Suppose  $|\lambda| = 1$  and  $\lambda$  is not a root of unity. As above there is a sequence of letters  $x_n \in X$  such that for each  $n$  either  $|\xi(g|_{x_1x_2\dots x_{n+1}})| > |\xi(g|_{x_1x_2\dots x_n})|$  or  $\xi(g|_{x_1x_2\dots x_{n+1}}) = \lambda\xi(g|_{x_1x_2\dots x_n})$ . In either case we have a contradiction with the fact that the action is finite-state.  $\square$

It is left to prove that Jordan cells for roots of unity have size 1. Let  $m$  be an integer number such that  $\varepsilon^m = 1$  for every root of unity  $\varepsilon$  from the spectrum of  $\phi$ . Then the spectrum of  $\phi^m$  consists of 1 and numbers less than 1 in modulus. The self-similar action  $(G, X^*)$  over the alphabet  $X$  induces the self-similar action  $(G, (X^m)^*)$  over the alphabet  $X^m$  of words of length  $m$  over  $X$ . Moreover, since the action  $(G, X^*)$  is finite-state then obviously the action  $(G, (X^m)^*)$  is also finite-state. Note that the virtual endomorphism  $\phi_v$  of the action  $(G, (X^m)^*)$  associated to a word  $v = x_1 \dots x_m \in X^m$  is the composition  $\phi_v = \phi_{x_1} \circ \dots \circ \phi_{x_m}$  of virtual endomorphisms of the action over  $X$ . In particular,  $\phi_{x_1 \dots x_1} = \phi^m$  and the action  $(G, (X^m)^*)$  corresponds to the pair  $(G, \phi^m)$ . If we know that the size of Jordan cells of  $\phi^m$  with eigenvalue 1 have size 1, then the same holds for  $\phi$  for roots of unity. Hence we can assume that all roots of unity in the spectrum of  $\phi$  are equal to 1.

Suppose there is a Jordan cell of  $\phi$  with eigenvalue 1 that has size greater than 1. Then there exist nonzero vectors  $v, u \in \mathcal{L}$  such that  $\phi(v) = v$  and  $\phi(u) = u + v$ . Since the matrix of  $\phi$  has rational entries, the vectors  $v$  and  $u$  can be chosen to have rational entries and we assume  $v, u \in \mathcal{L}_{\mathbb{Q}}$ . By Theorem 5.4.2 from [4] the group  $G$  contains a subgroup  $G_0$  of finite index such that  $\log G_0$  is a lattice in  $\mathcal{L}_{\mathbb{Q}}$ , i.e.,  $\log G_0$  is closed under addition and its span over  $\mathbb{Q}$  is equal to  $\mathcal{L}_{\mathbb{Q}}$ . Multiplying  $v$  and  $u$  by a suitable integer we can assume that they belong to  $\log G_0$ , and thus  $u + nv \in \log G_0$  for all  $n \in \mathbb{N}$ . Consider the element  $g = \exp(u) \in G$ . We get

$$\phi^n(g) = \phi^n(\exp(u)) = \exp(\phi^n(u)) = \exp(u + nv) \in G_0 \subset G,$$

and hence  $\phi^{n-1}(g) \in \phi^{-1}(G) = H = St_G(x_1)$  for all  $n \geq 1$ . Then the element  $g$  fixes the word  $x_1 x_1 \dots x_1$  ( $n$  times) and has the state  $g|_{x_1 x_1 \dots x_1} = \phi^n(g) = \exp(u + nv)$  for all  $n \geq 1$ . Since all elements  $u + nv$  are different, the element  $g$  is not finite-state. We got a contradiction.  $\square$

*Proof of Theorem 2.* If the spectral radius of  $\phi$  is less than 1, then the action is contracting and thus finite-state.

For the converse, it is sufficient to prove that if the Jordan normal form of  $\phi$  satisfies item 1 of the theorem and the spectrum of  $\phi$  contains a root of unity then there exists a non-finite-state action for  $(G, \phi)$ . As in the previous proof, we can assume that all roots of unity from the spectrum are equal to 1, and we find an element  $h \in H$  such that  $\phi(h) = h$ . The virtual endomorphism  $\phi|_{Z(H)} : Z(H) \rightarrow Z(G)$  has spectral radius less than 1. Let us choose a set of coset representatives  $D = \{h_1 = e, h_2, \dots, h_{d'}\}$  for  $Z(H)$  in  $Z(G)$ . If every element  $g \in Z(G)$  can be expressed as a product

$$g = h_{i_1} \phi^{-1}(h_{i_2} \phi^{-1}(\dots \phi^{-1}(h_{i_n}) \dots)) = h_{i_1} \phi^{-1}(h_{i_2}) \phi^{-2}(h_{i_3}) \dots \phi^{-n+1}(h_{i_n}) \quad (4)$$

for  $h_{i_j} \in D$ , then we take  $k > 1$  such that the set  $D^k = \{h_1^k = e, h_2^k, \dots, h_{d'}^k\}$  consists of coset representatives for  $Z(H)$  in  $Z(G)$  and replace  $D$  by  $D^k$ . In this case every product in (4) belongs to a proper subgroup  $Z(G)^k$  of  $Z(G)$ . We complete  $D$  to the set of coset representatives of  $H$  in  $G$  by elements  $h_{d'+1}, \dots, h_d$ . Replace the coset representative  $h_1 = e$  by the element  $h \in H$ . Let us prove that the associated self-similar action of the group  $G$  is not finite-state.

Take an element  $g \in Z(G)$  that cannot be expressed in the form (4), and consider the state of  $g$  at the word  $x_1 x_1 \dots x_1$  ( $n$  times):

$$g|_{x_1 x_1 \dots x_1} = \phi(h_{y_n}^{-1} \dots \phi(h_{y_2}^{-1} \phi(h_{y_1}^{-1} g h) h) \dots h) = \phi(h_{y_n}^{-1} \dots \phi(h_{y_2}^{-1} \phi(h_{y_1}^{-1} g)) \dots) h^n,$$

where  $g(x_1 x_1 \dots x_1) = y_1 y_2 \dots y_n$ . All elements  $h_{y_i}$  are taken from the set  $\{h, h_2, \dots, h_{d'}\}$ . The elements  $\phi(h_{y_j}^{-1} \dots \phi(h_{y_2}^{-1} \phi(h_{y_1}^{-1} g)) \dots)$  with all  $h_{y_k} \in \{h_2, \dots, h_{d'}\}$  belongs to the center  $Z(G)$ . We can move every element  $h_{y_i}$  that is equal to  $h$  to the right and include in the power  $h^n$ . Hence we can write the state as

$$g|_{x_1 x_1 \dots x_1} = \phi(h_{y_n}^{-1} \dots \phi(h_{y_2}^{-1} \phi(h_{y_1}^{-1} g)) \dots) h^m, \quad (5)$$



with all  $h_{y_i} \in \{e, h_2, \dots, h_d\}$  (here we replace every  $h_{y_i} = h$  by  $h_{y_i} = e$ ), and  $m$  is equal to the number of letters  $y_i$  not equal to  $x_1$ . As above, all the products  $\phi(h_{y_n}^{-1}) \dots \phi^{n-1}(h_{y_2}^{-1}) \phi^n(h_{y_1}^{-1}) \phi^n(g)$  belong to a compact subset of  $L$ , but also belong to the lattice  $G$ . Hence these products assume a finite number of values. Let us analyze the values of  $m$ .

Notice that  $g$  cannot stabilize the sequence  $x_1 x_1 \dots$ . Indeed, if  $g(x_1) = x_1$  then  $g|_{x_1} = \phi(h^{-1}gh) = \phi(g) \in Z(G)$ . Hence, if  $g(x_1 x_1 \dots) = x_1 x_1 \dots$  then  $\phi^n(g) \in Z(G)$  for all  $n \geq 1$ . It implies that there exists a non-trivial normal  $\phi$ -invariant subgroup of  $Z(G)$  and we get a contradiction with the faithfulness of the action. Suppose  $g$  changes only finitely many letters in the sequence  $x_1 x_1 \dots$ . Then  $g|_{x_1 x_1 \dots x_1}$  and  $g|_{x_1 x_1 \dots x_1} h^{-m}$  stabilize  $x_1 x_1 \dots$  for long enough word  $x_1 x_1 \dots x_1$ . At the same time

$$g|_{x_1 x_1 \dots x_1} h^{-m} = \phi(h_{y_n}^{-1} \dots \phi(h_{y_2}^{-1} \phi(h_{y_1}^{-1} g)) \dots) \in Z(G).$$

and we get that this element should be trivial. However in this case  $g$  can be expressed in the form (4), we get a contradiction with the choice of  $g$ .

Since  $g$  changes infinitely many letters in the sequence  $x_1 x_1 \dots$ , the number  $m$  in Equation (5) goes to infinity as the length of  $x_1 x_1 \dots x_1$  goes to infinity. The elements  $h^m$  are all different, and hence  $g$  has infinitely many states.  $\square$

### 3 Example

Consider the discrete Heisenberg group

$$G = \left\{ (x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\},$$

its subgroup  $H = \{(x, 2y, 2z) : x, y, z \in \mathbb{Z}\}$ , and the isomorphism  $\phi : H \rightarrow G$  given by  $\phi(x, y, z) = (x, y/2, z/2)$ . One can directly check that the  $\phi$ -core( $H$ ) is trivial, and every self-similar action for the pair  $(G, \phi)$  is faithful (we can also notice that  $\phi|_{Z(H)} : Z(H) \rightarrow Z(G)$  leads to the faithful self-similar action of  $(Z(G), \phi|_{Z(H)})$ , and hence the same holds for  $(G, \phi)$  by [2, Corollary 1]).

The matrix of  $\phi$  is diagonal with eigenvalues  $1, \frac{1}{2}, \frac{1}{2}$ , and Theorems 1 and 2 imply that there exist both finite-state and non-finite-state self-similar actions for the pair  $(G, \phi)$ . First, let us construct a finite-state action. Choose coset representatives  $D = \{(0, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1)\}$  and consider the associated self-similar action  $(G, X^*)$  over the alphabet  $X = \{1, 2, 3, 4\}$  given by Equation (1). The action of the generators  $a = (1, 0, 0)$  and  $b = (0, 1, 0)$  of the group satisfies the following recursions:

$$\begin{array}{llll} a(1v) = 1a(v) & a(2v) = 4a(v) & a(3v) = 3a(v) & a(4v) = 2(b^{-1}ab)(v) \\ b(1v) = 2v & b(2v) = 1b(v) & b(3v) = 4v & b(4v) = 3b(v) \end{array}$$

The elements  $a$  and  $b$  are finite-state, namely the states of  $a$  are  $a, b^{-1}ab, b^{-2}ab^2$  and the states of  $b$  are  $e, b$ . Hence the action  $(G, X^*)$  is finite-state.

Let us change the coset representatives and choose  $D' = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1)\}$ . Then the action of the generators of the group  $G$  satisfies the recursions

$$\begin{aligned} a(1v) &= 1a(v) & a(2v) &= 4a(v) & a(3v) &= 3a(v) & a(4v) &= 2(b^{-1}ab)(v) \\ b(1v) &= 2a(v) & b(2v) &= 1(a^{-1}b)(v) & b(3v) &= 4v & b(4v) &= 3b(v) \end{aligned}$$

In order to see that this action is not finite-state, let us look at the action of the element  $c = (0, 0, 1) = a^{-1}b^{-1}ab$ :

$$c(1v) = 3(a^2b^{-1}a^{-1}b)(v) \quad c(2v) = 4c(v) \quad c(3v) = 1a^{-1}(v) \quad c(4v) = 2c^2(v)$$

It is not difficult to deduce that all powers  $c^n$  for  $n \in \mathbb{N}$  are the states of  $c$  and hence the element  $c$  is not finite-state.

**Remark about growth of Schreier graphs.** Let  $G$  be a group with a finite generating set  $S$  and acting on a set  $M$ . The (simplicial) graph  $\Gamma(G, S, M)$  of the action is the graph with the set of vertices  $M$  and two points  $u, v \in M$  are connected by an edge if  $s(u) = v$  or  $s(v) = u$  for some  $s \in S$ . The connected component of  $\Gamma(G, S, M)$  around a point  $w \in M$  is called the orbital Schreier graph  $\Gamma_w(G, S)$ . The graph  $\Gamma_w(G, S)$  is the Schreier coset graph of  $G$  with respect to the stabilizer  $St_G(w)$ .

Every self-similar action  $(G, X^*)$  naturally extends to the action of  $G$  on the space  $X^\infty$  of left-infinite words  $x_1x_2\dots$  over the alphabet  $X$ . We get an uncountable family of orbital Schreier graphs  $\Gamma_w$  for  $w \in X^\infty$  of the action  $(G, X^\infty)$ . For every contracting self-similar action  $(G, X^\infty)$  all orbital Schreier graphs  $\Gamma_w$  have polynomial growth (see [7, Proposition 2.13.8]). Since nilpotent groups have polynomial growth, all their Schreier graphs also have polynomial growth. Hence the constructed above self-similar action of the Heisenberg group provides an example of a self-replicating finite-state and non-contracting self-similar action with orbital Schreier graphs of polynomial growth. The following problem seems to be interesting: characterize finitely-generated finite-state self-similar groups (i.e., groups generated by finite automata), whose all orbital Schreier graphs  $\Gamma_w$  have polynomial growth. More generally, given a finitely generated group  $G$  and a nested sequence  $\{H_n\}_{n \geq 1}$  of subgroups of finite index in  $G$  with trivial intersection  $\cap_{n \geq 1} H_n = \{e\}$ , consider the action of  $G$  on the coset tree of  $\{H_n\}_{n \geq 1}$ . In what cases the orbital Schreier graphs of the action of  $G$  on the boundary of the coset tree have polynomial growth?

We also want to remark the following property of the finite-state self-similar action of the Heisenberg group constructed above. The action is not free on the orbits of points from  $\{1, 3\}^\infty$ , the stabilizer  $St_G(w)$  for  $w \in \{1, 3\}^\infty$  is the infinite cyclic group, and the corresponding Schreier graphs  $\Gamma_w$  have polynomial growth of degree 3. The action on the other orbits is free, and the corresponding Schreier graphs  $\Gamma_w$  have polynomial growth of degree 4. Hence we get an example of a finite-state self-similar group with transitive action on  $X^n$  for all  $n \in \mathbb{N}$ , which has orbital Schreier graphs  $\Gamma_w$  with different growth.



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